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A unified matrix formulation for the zone method: a stochastic approach

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Abstract—A new approach based on the Markov chain theory is utilized to derive three explicit matrix relations for the total exchange areas of the zone method. These relations are identical to those published earlier. The same method is used to derive a single explicit matrix relation for the total exchange areas. This expression is so general that it covers the same function as the aforementioned relations.

1. INTRODUCTION

THE RESULTING equations of a differential formulation for radiation in enclosures with gas participation are extremely complex to be solved analytically. Therefore, when the geometry under consideration is complicated, the zone method suggested by Hottel *et al.* [1, 2] is generally adopted to analyze radiative exchange in enclosures. In this method the space for which the radiation exchange has to be calculated is divided into a number of zones. Each zone is assumed to have a uniform temperature and constant thermal properties. This method leads to a set of non-linear algebraic equations which can be solved numerically for temperatures and heat fluxes by means of Ness' rapidly converging method [3].

Recently, Noble [4] developed a set of explicit matrix relations for the calculation of total exchange areas. The use of these relations reduces the computation effort considerably. Noble's formulations are particularly useful for the development of a highly efficient subroutine for furnace design calculations.

The present work employs a stochastic approach, which is based on the Markov chain theory [5, 6], to calculate total exchange areas of an enclosure with diffusely reflecting walls, involving isotropically absorbing–emitting and scattering media. The Markov chain theory has been shown to be a powerful tool for modeling chemical reactor systems [7–10]. Recently, Naraghi and Chung [11] used Markov chain theory to determine radiation exchange in enclosures with non-participating media. The purpose of this work is to further extend the application of Markov chain theory to radiative transfer involving participating medium. An extensive application of stochastic approach to radiative transfer may be found in reference [12].

One of the existing stochastic methods of solution is known as the Monte Carlo method which was developed by Howell and Perlmutter [13, 14]. Recently the Monte Carlo method was employed to improve the zone method by Vercammen and Froment [15].

In this study, the stochastic approach will be utilized to derive three explicit matrix formulations for total exchange areas. These relations are identical to those obtained by Noble [4]. Furthermore, a simple explicit matrix relation is derived which performs the function of all three of the above matrix formulations simultaneously. This relation is referred to as the unified matrix formulation. The application of this relation requires much less computational effort as compared to the conventional zone method and the Noble formulation.

2. STOCHASTIC PROCESS AND MARKOV CHAIN THEORY

A stochastic process is a description of random phenomenon changing with respect to time. It may be thought of as a set of random variables x_t depending on a parameter t with $t \in T$. Here, the time variable t is discrete and is in terms of reflection or scattering numbers. Hence, the stochastic process used in this work is a set of random variables x_n depending on variable $n \in N$. In short notation, it is expressed in the form of $X = \{x_n, n \in N\}$ or $X = \{x_0, x_1, x_2, \ldots\}$ in which the x_ns assume numerical values equal to the zone number of the enclosure. For example, if an enclosure consists of two surface zones s_1, s_2 and three gas zones g_1, g_2, g_3 then the stochastic process $X = \{s_1, g_2, g_1, s_2, g_1, g_3, g_1, s_2, \ldots\}$ represents a process whereby a radiative energy bundle initially emitted from surface zone s_1, is then scattered from gas zones g_2 and g_1, reflected from surface s_2, scattered from gas zones g_1, g_3, and g_1, reflected from $s_2, \ldots$, and finally is absorbed by one of the participating zones. Hence, stochastic processes can be used to represent radiation processes in enclosures.

A Markov chain is a stochastic process in which the following property holds

$$p\{x_n | x_0, x_1, x_2, \ldots, x_{n-1}\} = p\{x_n | x_{n-1}\} \qquad (1)$$

or that the probability of an event, x_n, is only dependent on the immediately preceding event, x_{n-1}. The radiation transport process in enclosures with diffusely reflecting surfaces and an isotropically scattering gas forms a Markov chain. This is due to the fact that the future of the radiative energy bundle after diffuse

NOMENCLATURE

A_i area of the surface i [m^2]
A $[A_i\delta_{i,j}]$, diagonal matrix with elements A_i
A^{-1} inverse of matrix A
A' $I-R'$ equivalent absorption matrix
B equivalent area matrix defined by equation (20)
E $[\varepsilon_i\delta_{i,j}] = [\alpha_i\delta_{i,j}]$ emissivity or absorptivity matrix
I $[\delta_{i,j}]$, identity matrix
k_a absorption coefficient [m^{-1}]
k_s scattering coefficient [m^{-1}]
k_t total extinction coefficient [m^{-1}]
L spacing between parallel plates
m number of gas zones
n number of surface zones
$\overline{g_ig_j}$ direct exchange area from gas zone i to gas zone j [m^2]
$\overline{gg}$ $[\overline{g_ig_j}]$
$\overline{G_iG_j}$ total exchange area from gas zone i to gas zone j [m^2]
$\overline{GG}$ $[\overline{G_iG_j}]$
$\overline{G_iS_j}$ total exchange area from gas zone i to surface zone j [m^2]
$\overline{GS}$ $[\overline{G_iS_j}]$
$p_{i,j}$ transition probability (probability that the radiative energy bundle goes from zone i to zone j)
P $[p_{i,j}]$ transition probability matrix
P_k transition probability for the kth transition
$p_{g_ig_j}$ probability that radiation reaches gas zone j having been emitted from gas zone i, allowing multiple scattering but excluding wall reflections
P_{gg} $[p_{g_ig_j}]$
$p_{s_ig_j}$ probability that radiation reaches gas zone j, having been emitted from surface zone i, allowing multiple scattering but excluding wall reflections
P_{sg} $[p_{s_ig_j}]$
$p^{(k)}_{s_is_j}$ probability that radiation emitted from surface i reaches surface zone j after being scattered exactly k times
$P^{(k)}_{ss}$ $[p^{(k)}_{s_is_j}]$
$p_{s_is_j}$ probability that radiation emitted from surface zone i reaches surface zone j, allowing multiple scattering but excluding wall reflections
P_{ss} $[p_{s_is_j}]$
$p_{z_iz_j}$ probability that radiation emitted from zone i reaches zone j, allowing multiple scattering but excluding wall reflections
R $[\rho_i\delta_{i,j}]$, matrix of reflectivities
R' equivalent reflection matrix defined by equation (19)
$\overline{s_ig_j}$ direct exchange area from surface zone i to gas zone j [m^2]
$\overline{sg}$ $[\overline{s_ig_j}]$
$\overline{S_iG_j}$ total exchange area from surface zone i to gas zone j [m^2]
$\overline{SG}$ $[\overline{S_iG_j}]$
$\overline{s_is_j}$ direct exchange area from surface zone i to surface zone j [m^2]
$\overline{ss}$ $[\overline{s_is_j}]$
$\overline{S_iS_j}$ total exchange area from surface zone i to surface zone j [m^2]
$\overline{SS}$ $[\overline{S_iS_j}]$
V_i volume of gas zone i [m^3]
V $[V_i\delta_{i,j}]$, volume matrix
x random variable
X stochastic process
$\overline{zz}$ partitioned matrix defined by equation (17)
$\overline{ZZ}$ partitioned matrix defined by equation (18).

Greek symbols
α_i absorptivity of surface zone i
ε_i emissivity of surface zone i
ρ_i reflectivity of surface zone i
τ_0 optical thickness between parallel plates
ω_0 k_s/k_t, albedo for scattering.

Subscripts
s designates surface zone
g designates gas zone.

Superscripts
* denotes dimensionless exchange area, e.g. $\overline{ss}^* = A^{-1}\overline{ss}$, $\overline{sg}^* = A^{-1}\overline{sg}$, $\overline{gs}^* = [4k_tV]^{-1}\overline{gs}$, $\overline{gg}^* = [4k_tV]^{-1}\overline{gg}$, $\overline{GG}^* = [4k_tV]^{-1}\overline{GG}$, $\overline{SG}^* = [EA]^{-1}\overline{SG}$ and $\overline{SS}^* = [EA]^{-1}\overline{SS}$.

reflection and isotropic scattering is independent of its past history. The probability that the radiative energy bundle, having been emitted from zone i, reaches zone j is called the transition probability. In short notation the transition probability can be written as

$$p\{x_n = j | x_{n-1} = i\} = p_{i,j}. \tag{2}$$

Note that the transition probability represented by the above equation does not change with time (n). The Markov chains with this property are called time-homogeneous Markov chains. For a k-state time-homogeneous Markov chain, the transition probability matrix is

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,k} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ p_{k,1} & p_{k,2} & \cdots & p_{k,k} \end{bmatrix}. \tag{3}$$

An m-step transition probability, $p_{i,j}^{(m)}$, is defined by

$$p\{x_m = j | x_0 = i\} = p_{i,j}^{(m)} \tag{4}$$

i.e. the probability that the process reaches state j in m transitions having initially begun from state i. For a Markov chain we can write [6]

$$[p_{i,j}^{(m)}] = P_1 P_2 P_3 \ldots P_m \tag{5}$$

where $P_1, P_2, P_3, \ldots, P_m$ are the transition probability matrices for the first, second, third, and . . . , mth transitions, respectively. If these transition probabilities are equal, i.e. $P_1 = P_2 = P_3 = \cdots = P_m = P$ then

$$[p_{i,j}^{(m)}] = P^m. \tag{6}$$

Equations (5) and (6) can be used to calculate m-step transition probabilities in terms of single-step transition probabilities.

3. ANALYSIS

The property of the Markov chain given by equation (5) will be used to calculate the probability that radiation, having been emitted from zone z_i of an enclosure, will be absorbed by zone z_j in the same enclosure after multiple reflection and/or scattering. These probabilities are identical to the dimensionless total exchange area $\overline{Z_iZ_j}^*$.

The dimensional and dimensionless total exchange areas are related to each other according to the following relations:

$\overline{Z_iZ_j} = A_i\varepsilon_i\overline{Z_iZ_j}^*$ when z_i is an emitting surface

$\overline{Z_iZ_j} = 4K_tV_i\overline{Z_iZ_j}^*$ when z_i is an emitting volume.

By definition, the dimensional total exchange area $\overline{Z_iZ_j}$ is the ratio of the radiation energy emitted from zone z_i, which is absorbed by zone z_j (directly or after reflection from other zones) to the total hemispherical emissive power of zone z_i. We define, $p_{z_iz_i}$ as the probability that radiation emitted from zone z_i reaches zone z_j, allowing multiple scattering but without wall reflection. The relations between $p_{z_iz_j}$s and direct exchange areas will be developed in the following section. Using the transition probability matrix, the total exchange areas are calculated.

3.1. *Three matrix formulations*

In this section three separate matrix relations for the total exchange areas will be derived. The transition probabilities are identical to the dimensionless direct exchange areas, i.e. $\overline{ss}^* = A^{-1}\overline{ss}$, $\overline{sg}^* = A^{-1}\overline{sg}$, $\overline{gs}^* = [4k_tV]^{-1}\overline{gs}$ and $\overline{gg}^* = [4k_tV]^{-1}\overline{gg}$. This is because the dimensionless direct exchange area, $\overline{z_iz_j}^*$, is the fraction of energy emitted from zone z_i that reaches zone z_j by direct radiation, without reflection or scattering from other zones; this quantity is identical to the probability that an energy bundle emitted from zone z_i reaches zone z_j by direct radiation. For convenience, the same notation as was used in [4] will be employed herein.

We first calculate the probability that radiation emitted from surface i reaches surface j without wall reflection, $P_{s_is_j}$. The radiative energy bundle is allowed to be scattered by the gas zones any number of times (see Fig. 1). We define $P_{ss}^{(k)} = [p_{s_is_j}^{(k)}]$, where $p_{s_is_j}^{(k)}$ is the probability that radiation emitted from surface i reaches surface j after being scattered exactly k times. Since the radiation process in an isotropically scattering gas forms a Markov chain, using equation (5), we can write

$$P_{ss}^{(0)} = \overline{ss}^*$$

$$P_{ss}^{(1)} = \overline{sg}^*\omega_0\overline{gs}^*$$

$$P_{ss}^{(2)} = \overline{sg}^*\omega_0\overline{gg}^*\omega_0\overline{gs}^*$$

$$P_{ss}^{(3)} = \overline{sg}^*\omega_0\overline{gg}^*\omega_0\overline{gg}^*\omega_0\overline{gs}^* = \overline{sg}^*(\omega_0\overline{gg}^*)^2\omega_0\overline{gs}^*$$

$$\vdots$$

$$P_{ss}^{(k)} = \overline{sg}^*\omega_0\overline{gg}^*\omega_0\overline{gg}^*\ldots\omega_0\overline{gs}^*$$

$$= \overline{sg}^*(\omega_0\overline{gg}^*)^{k-1}\omega_0\overline{gs}^*$$

$$\vdots$$

$$P_{ss} = [p_{s_is_j}] = \sum_{k=0}^{\infty} P_{ss}^{(k)} = \overline{ss}^* + \overline{sg}^*$$

$$\times [I + \omega_0\overline{gg}^* + (\omega_0\overline{gg}^*)^2 + \cdots]\omega_0\overline{gs}^*$$

$$= \overline{ss}^* + \overline{sg}^*[I - \omega_0\overline{gg}^*]^{-1}\omega_0\overline{gs}^*. \tag{7}$$

Note that the convergence of the above series is assured because

$$\|\omega_0\overline{gg}^*\| < 1$$

where $\|\cdot\|$ is the norm of the matrix. It should be emphasized that the above formulation is valid only for an enclosure with diffuse and isotropic scattering media, otherwise, the Markov chain property is not satisfied. Now we proceed to calculate the matrices $\overline{SS}^*$ and $\overline{SS}$. We define $\overline{SS}^{*(k)} = [\overline{S_iS_j}^{*(k)}]$ where $\overline{S_iS_j}^{*(k)}$ is

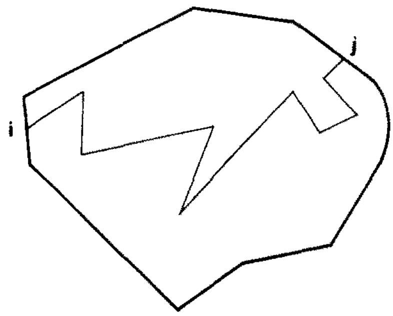

FIG. 1. A possible route that radiative energy bundle could travel between two surfaces (no wall reflection, allowing gas scattering).

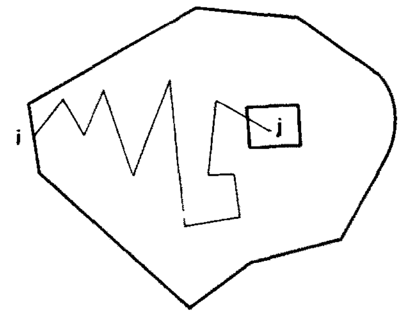

FIG. 3. A possible route that radiative energy bundle could travel between a surface and a gas zone (no wall reflection, allowing gas scattering).

the probability that radiation emitted from surface i is absorbed by surface j in exactly k reflections from surfaces, allowing multiple scattering (see Fig. 2). The multiple scattering is already included in transition probabilities P_{ss}. Based on the Markovian property of diffusely reflecting radiation and equation (5), we can write

$$\overline{SS}^{*(1)} = P_{ss}E$$

$$\overline{SS}^{*(2)} = P_{ss}RP_{ss}E$$

$$\overline{SS}^{*(3)} = P_{ss}RP_{ss}RP_{ss}E = (P_{ss}R)^2P_{ss}E$$

$$\vdots$$

$$\overline{SS}^{*(k)} = P_{ss}RP_{ss}RP_{ss}R\ldots P_{ss}E = (P_{ss}R)^{k-1}P_{ss}E.$$

$$\vdots$$

The dimensionless total exchange area $\overline{SS}^*$ is the summation of all $\overline{SS}^{*(k)}$s, i.e.

$$\overline{SS}^* = \sum_{k=1}^{\infty} \overline{SS}^{*(k)} = [I+(P_{ss}R)+(P_{ss}R)^2+\cdots]P_{ss}E$$
$$= [I-P_{ss}R]^{-1}P_{ss}E. \tag{8}$$

Substituting for P_{ss} from equation (7) we obtain

$$\overline{SS}^* = [I-\{\overline{ss}^*+\omega_0\overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}\overline{gs}^*\}R]^{-1}$$
$$\cdot\{\overline{ss}^*+\omega_0\overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}\overline{gs}^*\}E. \tag{9}$$

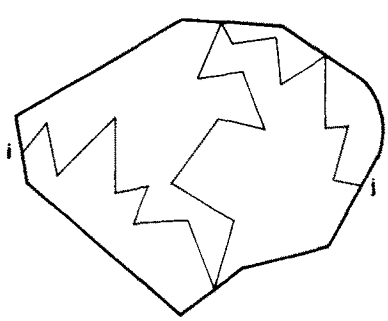

FIG. 2. A possible route that radiative energy bundle could travel between two surfaces (allowing wall reflections and gas scattering).

The dimensional version of the above expression is

$$\overline{SS} = EA[A-\{\overline{ss}+\omega_0\overline{sg}[4k_tV-\omega_0\overline{gg}]^{-1}\overline{gs}\}R]^{-1}$$
$$\cdot\{\overline{ss}+\omega_0\overline{sg}[4k_tV-\omega_0\overline{gg}]^{-1}\overline{gs}\}E. \tag{10}$$

Next we calculate the total exchange area between the surface and gas zones, $\overline{SG}^*$ or $\overline{SG}$. We define $p_{s_ig_j}$ as the probability that the radiative energy bundle reaches gas zone g_j, having been emitted from surface s_i, allowing multiple scattering but excluding wall reflections (see Fig. 3). Based on the Markovian property of the radiation the processes in isotropically scattering media and equation (5), we can write

$$P_{sg} = [p_{s_ig_j}]$$
$$= \overline{sg}^*+\overline{sg}^*\omega_0\overline{gg}^*+\overline{sg}^*\omega_0\overline{gg}^*\omega_0\overline{gg}^*+\cdots$$
$$= \overline{sg}^*[I+\omega_0\overline{gg}^*+(\omega_0\overline{gg}^*)^2+\cdots]$$
$$= \overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}. \tag{11}$$

The radiation initially emitted from the surface zone s_i can reach gas zone g_j via scattering without wall reflection or with a number of wall reflections. Based on the Markovian property of radiation process in enclosures diffusely reflecting walls and equation (5), we can write

$$\overline{SG}^* = P_{sg}(1-\omega_0)+P_{ss}RP_{sg}(1-\omega_0)$$
$$+P_{ss}RP_{ss}RP_{sg}(1-\omega_0)+\cdots$$
$$= [I+P_{ss}R+(P_{ss}R)^2+\cdots]P_{sg}(1-\omega_0)$$
$$= [I-P_{ss}R]^{-1}P_{sg}(1-\omega_0).$$

Upon substitution of P_{ss} and P_{sg} from equations (7) and (11), respectively, we obtain

$$\overline{SG}^* = [I-\{\overline{ss}^*+\omega_0\overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}\overline{gs}^*\}R]^{-1}$$
$$\cdot\overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}(1-\omega_0). \tag{12}$$

The dimensional form of the above equation is

$$\overline{SG} = EA[A-\{\overline{ss}+\omega_0\overline{sg}[4k_tV-\omega_0\overline{gg}]^{-1}\overline{gs}\}R]^{-1}$$
$$\cdot\overline{sg}[4k_tV-\omega_0\overline{gg}]^{-1}[4k_tV](1-\omega_0). \tag{13}$$

By the reciprocal rule, $\overline{GS}$ can be obtained from the transpose of $\overline{SG}$.

The radiation exchange between two gas zones consists of two parts, (1) radiation between two gas zones without wall reflections and with gas scattering (see Fig. 4(a)), and (2) radiation between two gas zones with a number of wall reflections and gas scattering (see Fig. 4(b)).

First, we calculate the probability that radiation reaches gas zone g_j having been emitted from gas zone g_i without wall reflections. Based on the Markovian property of gas scattering we can write

$$\begin{aligned} P_{gg} &= [p_{g_i g_j}] \\ &= \overline{gg}^* + \overline{gg}^*\omega_0\overline{gg}^* + \overline{gg}^*\omega_0\overline{gg}^*\omega_0\overline{gg}^* + \cdots \\ &= \overline{gg}^*[I-\omega_0\overline{gg}^*]^{-1}. \end{aligned} \tag{14}$$

The total exchange area between two gas zones is

$$\begin{aligned} \overline{GG}^* &= P_{gg}(1-\omega_0) + P_{gs}RP_{sg}(1-\omega_0) \\ &\quad + P_{gs}RP_{ss}RP_{sg}(1-\omega_0) \\ &\quad + P_{gs}RP_{ss}RP_{ss}RP_{sg}(1-\omega_0) + \cdots \\ &= P_{gg}(1-\omega_0) + P_{gs}R[I-P_{ss}R]^{-1}P_{sg}(1-\omega_0). \end{aligned}$$

Substituting the expressions for P_{gg}, P_{gs}, P_{ss} and P_{sg} we arrive at

$$\begin{aligned} \overline{GG}^* &= \overline{gg}^*[I-\omega_0\overline{gg}^*]^{-1}(1-\omega_0) \\ &\quad + [I-\omega_0\overline{gg}^*]^{-1}\overline{gs}^*R \\ &\quad \cdot [I-\{\overline{ss}^* + \omega_0\overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}\overline{gs}^*\}R]^{-1} \\ &\quad \cdot \overline{sg}^*[I-\omega_0\overline{gg}^*]^{-1}(1-\omega_0). \end{aligned} \tag{15}$$

The dimensional form of the above expression is

$$\begin{aligned} \overline{GG} &= (1-\omega_0)^2[4k_tV]\overline{gg}[4k_tV-\omega_0\overline{gg}]^{-1} \\ &\quad + (1-\omega_0)^2[4k_tV][4k_tV-\omega_0\overline{gg}]^{-1}\overline{gs}R \\ &\quad \cdot [A-\{\overline{ss}+\omega_0\overline{sg}[4k_tV-\omega_0\overline{gg}]^{-1} \\ &\quad \cdot \overline{gs}\}R]^{-1}\overline{sg}\cdot[4k_tV-\omega_0\overline{gg}]^{-1}[4k_tV]. \end{aligned} \tag{16}$$

The above three formulations are identical to those derived by Noble [4] using a radiosity approach. Appendix A demonstrates an application of equations (10), (13) and (16) to a problem which deals with radiation between two infinitely long parallel plates with an absorbing–emitting and scattering medium between them.

3.2. *Unified matrix formulation*

In this section a single matrix relation will be derived to give the total exchange areas in terms of direct exchange areas in one expression. We define $\overline{zz}$ and $\overline{ZZ}$ matrices as

$$\overline{zz} = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \left[\begin{array}{c|c} \overline{ss} & \overline{sg} \\ \hline \overline{gs} & \overline{gg} \end{array}\right] & \begin{array}{c} n \\ m \end{array} \end{array} \tag{17}$$

and

$$\overline{ZZ} = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \left[\begin{array}{c|c} \overline{SS} & \overline{SG} \\ \hline \overline{GS} & \overline{GG} \end{array}\right] & \begin{array}{c} n \\ m \end{array} \end{array} \qquad 18)$$

and an equivalent reflection matrix as

$$R' = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \left[\begin{array}{ccc|ccc} \rho_{s_1} & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & \rho_{s_n} & & & \\ \hline & & & \omega_0 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \omega_0 \end{array}\right] & \begin{array}{c} n \\ \\ \\ m \end{array} \end{array} \tag{19}$$

where n is the total number of surface zones and m the total number of volume zones. The absorption matrix is

$$A' = I - R'.$$

We also define an equivalent area matrix

$$B = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \left[\begin{array}{cccc|ccc} A_1 & & & & & & \\ & A_2 & & 0 & & 0 & \\ & 0 & \ddots & & & & \\ & & & A_n & & & \\ \hline & & & & 4k_tV_1 & & \\ & 0 & & & & \ddots & 0 \\ & & & & 0 & & 4k_tV_m \end{array}\right] & \begin{array}{c} n \\ \\ \\ m \end{array} \end{array}. \tag{20}$$

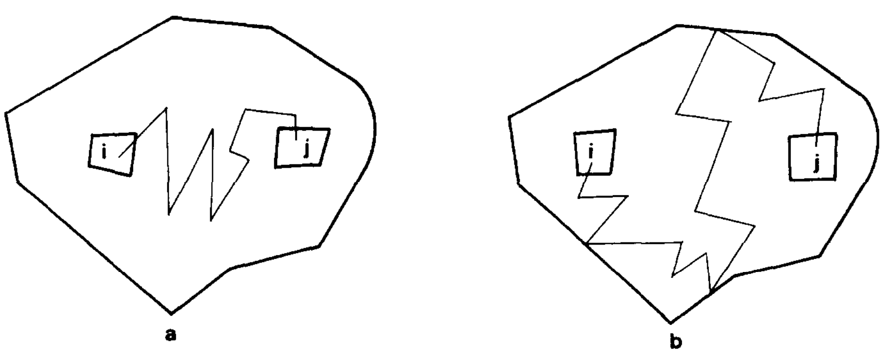

FIG. 4. Possible routes that radiative energy bundle could travel between two gas zones, (a) no wall reflection, allowing gas scattering, (b) wall reflection plus gas scattering.

The dimensionless direct exchange area matrix is

$$\overline{zz}^* = B^{-1}\overline{zz}.$$

Entries of matrix $\overline{zz}^*$ are identical to the probabilities that energy bundles emitted from zone z_i reached zone z_j by direct radiation. Hence, it is the transition probability matrix.

Based on the Markovian property of radiation processes in enclosures with diffusely reflecting walls and isotropically scattering media we can write

$$\begin{aligned}\overline{ZZ}^* &= \overline{zz}^*A' + \overline{zz}^*R'\overline{zz}^*A' + \overline{zz}^*R'\overline{zz}^*R'\overline{zz}^*A' + \cdots \\ &= [I - \overline{zz}^*R']^{-1}\overline{zz}^*A' \qquad (21)\end{aligned}$$

where $\overline{zz}^*A'$, $\overline{zz}^*R'\overline{zz}^*A'$, $\overline{zz}^*R'\overline{zz}^*R'\overline{zz}^*A' \ldots$ give the probabilities that energy bundles are being absorbed in one, two, three ... transitions, respectively. The dimensional form of equation (21) is

$$\overline{ZZ} = A'B[B - \overline{zz}R']^{-1}\overline{zz}A'. \qquad (22)$$

The above equation simultaneously performs the same function of the three explicit formulations derived in the previous section. Appendix B demonstrates the application of the above formulation to the same sample problem in Appendix A. It is seen that the results obtained using equation (22) are identical to those obtained by equations (10), (13) and (16) together.

4. CONCLUSION

A new stochastic approach for analyzing Hottel's zone method is developed. This method provides explicit matrix relations for the total exchange areas. The unified formulation presented in this study performs the same function as that of three existing formulations. Use of equation (22) for computer coding greatly reduces the program size and programming efforts.

The present approach can be further extended to radiative exchange in enclosures with bidirectionally reflecting surfaces and anisotropically scattering media. A preliminary analysis is given in [16] on the basis of multiple Markov chain theory [6].

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APPENDIX A

Consideration is given to two infinitely long parallel plates with an absorbing–emitting and scattering medium between them. The optical thickness between the plates is $\tau_0 = K_t L = 1$; the albedo for scattering $\omega_0 = 0.5$ and the emissivity of the surfaces are $\varepsilon_1 = 0.3$ and $\varepsilon_2 = 0.7$. The medium is divided into three zones. The direct exchange areas can be obtained from the formulations given in reference [2], they are

$$\overline{ss} = \begin{bmatrix} 0 & 0.21938 \\ 0.21938 & 0 \end{bmatrix} \qquad (A1)$$

$$\overline{sg} = \begin{bmatrix} 0.43021 & 0.22158 & 0.12883 \\ 0.12883 & 0.22158 & 0.43021 \end{bmatrix} \qquad (A2)$$

$$\overline{gs} = \overline{sg}^{T} \qquad (A3)$$

and

$$\overline{gg} = \begin{bmatrix} 0.47291 & 0.20864 & 0.09275 \\ 0.20864 & 0.47291 & 0.20864 \\ 0.09275 & 0.20864 & 0.47291 \end{bmatrix}. \qquad (A4)$$

Application of equations (10), (13) and (16) yields

$$\overline{SS} = \begin{bmatrix} 0.01620 & 0.07617 \\ 0.07617 & 0.10631 \end{bmatrix} \qquad (A5)$$

$$\overline{SG} = \begin{bmatrix} 0.09884 & 0.06359 & 0.04521 \\ 0.12962 & 0.15849 & 0.22942 \end{bmatrix} \quad \text{(A6)}$$

$$\overline{GS} = \overline{SG} \quad \text{(A7)}$$

and

$$\overline{GG} = \begin{bmatrix} 0.22434 & 0.13115 & 0.08273 \\ 0.13115 & 0.19644 & 0.11701 \\ 0.08273 & 0.11701 & 0.19231 \end{bmatrix}. \quad \text{(A8)}$$

The total exchange shown by equations (A5), (A6) and (A8) are obtained by using three tedious matrix formulations. As seen in Appendix B, the same total exchange areas will be calculated using one simpler formulation.

APPENDIX B

The same problem introduced in Appendix A will be solved using the unified matrix given by equation (22). The direct exchange area matrix is, from equation (17),

$$\overline{zz} = \left[\begin{array}{cc|ccc} 0.00000 & 0.21938 & 0.43021 & 0.22158 & 0.12883 \\ 0.21938 & 0.00000 & 0.12883 & 0.22158 & 0.43021 \\ \hline 0.43021 & 0.12883 & 0.47291 & 0.20864 & 0.09275 \\ 0.22158 & 0.22158 & 0.20864 & 0.47291 & 0.20864 \\ 0.12883 & 0.43021 & 0.09275 & 0.20864 & 0.47291 \end{array}\right]. \quad \text{(B1)}$$

Based on equation (19) the equivalent reflection matrix is

$$R' = \left[\begin{array}{cc|ccc} 0.7 & 0 & & 0 & \\ 0 & 0.3 & & & \\ \hline & & 0.5 & & 0 \\ & 0 & & 0.5 & \\ & & 0 & & 0.5 \end{array}\right] \quad \text{(B2)}$$

$$A' = I - R'.$$

From equation (20) the matrix B is given by

$$B = \left[\begin{array}{cc|ccc} 1 & 0 & & 0 & \\ 0 & 1 & & & \\ \hline & & 4/3 & & 0 \\ & 0 & & 4/3 & \\ & & 0 & & 4/3 \end{array}\right]. \quad \text{(B3)}$$

Application of equation (22) leads to

$$\overline{ZZ} = \left[\begin{array}{cc|ccc} 0.01620 & 0.07617 & 0.09884 & 0.06359 & 0.04521 \\ 0.07617 & 0.10631 & 0.12962 & 0.15849 & 0.22942 \\ \hline 0.09884 & 0.12962 & 0.22434 & 0.13115 & 0.08273 \\ 0.06359 & 0.15849 & 0.13115 & 0.19644 & 0.11701 \\ 0.04521 & 0.22942 & 0.08273 & 0.11701 & 0.19231 \end{array}\right]. \quad \text{(B4)}$$

The partitions of $\overline{ZZ}$ matrix given by equation (B4) represent $\overline{SS}$, $\overline{SG}$, $\overline{GS}$ and $\overline{GG}$ matrices which are identical to equations (A5)–(A8), respectively.

UNE FORMULATION MATRICIELLE UNIFIE POUR LA METHODES DES ZONES: UNE APPROCHE STOCHASTIQUE

Résumé—Une nouvelle approche basée sur la théorie de Markov est utilisée pour dériver trois relations matricielles explicites pour les aires d'échange de la méthode des zones. Ces relations sont identiques à celles publiées précédemment. La même methode est utilisée pour obtenir une relation matricielle explicite unique pour les aires d'échange total. Cette expression est si générale qu'elle couvre la même fonction que les relations ci-dessus mentionnées.

VERALLGEMEINERTE MATRIXBEZIEHUNG FÜR DAS ZONENVERFAHREN: EINE STOCHASTISCHE NÄHERUNG

Zusammenfassung—Ein neues Näherungsverfahren, das auf der Markov-Theorie beruht, wird dazu verwendet, drei explizite Matrix-Beziehungen für die Gesamtaustauschflächen des Zonenverfahrens zu entwickeln. Diese Beziehungen sind mit den früher veröffentlichten identisch. Dasselbe Verfahren wird dazu verwendet, eine einzelne Matrixbeziehung für die Gesamtaustauschflächen zu entwickeln. Dieser Ausdruck ist so allgemein, daß er die zuvor erwähnten Beziehungen mit abdeckt.

УНИФИЦИРОВАННАЯ МАТРИЧНАЯ ФОРМУЛИРОВКА В ЗОНАЛЬНОМ МЕТОДЕ: СТОХАСТИЧЕСКИЙ ПОДХОД

Аннотация—Новый метод, основанный на теории цепей Маркова, использован для вывода трех матричных соотношений в явном виде для определения суммарных поверхностей переноса в зональном методе. Эти соотношения идентичны опубликованным ранее. Этим же методом получено одно матричное соотношение в явном виде для определения суммарных поверхностей обмена. Выражение носит настолько общий характер, что может использоваться вместо указанных соотношений.